

A DUAL BRAID MONOID FOR THE FREE GROUP

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ABSTRACT. We construct a quasi-Garside monoid structure for the free group. This monoid should be thought of as a *dual braid monoid* for the free group, generalising the constructions by Birman-Ko-Lee and by the author of new Garside monoids for Artin groups of spherical type. Conjecturally, an analog construction should be available for arbitrary Artin groups and for braid groups of well-generated complex reflection groups.

This article continues the exploration of the theory of Artin groups and generalised braid groups from the new point of view introduced by Birman-Ko-Lee in [BKL] for the classical braid group on n strings. In [B1], we generalised their construction to Artin groups of spherical type. In the current article, we study the case of the free group, which is the Artin group associated with the universal Coxeter group. The formal analogs of the main statements in [B1] turn out to be elementary consequences of classical material (some of which was known to Hurwitz and Artin). In an attempt to interpolate some recent generalisations of the dual monoid construction (by Digne for the Artin group of type \widetilde{A}_n , [D]; by Corran and the author for the braid group of the complex reflection group $G(e, e, n)$, [BC]), we propose two conjectures describing properties of a generalised dual braid monoid, in the contexts of

- (a) arbitrary Artin groups and
- (b) braid groups of well-generated finite complex reflection groups.

This would provide the first uniform combinatorial approach to these objects. The initial motivation for the current work was to understand the situation (b) from a natural geometric viewpoint; the conjectures about complex reflection groups will be studied in the sequel [B2], answering some questions raised in [BMR].

1. HURWITZ ACTION

For any positive integer n , the “usual” braid group is the abstractly presented group

$$B_n := \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \rangle.$$

In the problems we are interested in, two “braid groups” simultaneously come into play: this “usual” braid group, and the Artin group associated with a Coxeter system (or the generalised braid group associated with a complex reflection group). Except in the final conjectures, this Artin group will be the free group.

Let G be a group. For any sequence $(g_1, \dots, g_n) \in G^n$, set

$$\sigma_i \cdot (g_1, \dots, g_n) := (g_1, \dots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \dots, g_n).$$

This article is the fruit of an inspiring visit to KIAS (Seoul) in June 2003. I thank Sang Jin Lee for his hospitality, for stimulating discussions and for important suggestions.

It is straightforward (and well-known) that this assignment extends to a left-action of B_n on G^n .

Definition 1.1. *This action is called Hurwitz action of B_n on G^n .*

This action can be viewed as a particular example of a more general construction, where the important property of G^n is that it is an *automorphic set* (in the sense of [Br]) or equivalently a *rack* (in the sense, for example, of [DDRW]).

In [Br], Brieskorn considers several problems about braid group actions on automorphic sets. One of these problems is to characterise orbits. A very naive invariant of Hurwitz action is the product

$$\pi : (g_1, \dots, g_n) \mapsto g_1 \dots g_n.$$

We will be interested in situations where $\pi^{-1}(g)$ is a single Hurwitz orbit, for a specific $g \in G$.

2. NON-CROSSING LOOPS

In all this section, we fix $n + 1$ distinct points x_0, \dots, x_n in \mathbb{C} . The complex line is endowed with an orientation called “positive” or “direct”.

We set

$$F_n := \pi_1(\mathbb{C} - \{x_1, \dots, x_n\}, x_0).$$

This group is isomorphic to a (“the”) free group on n generators, but its geometric definition gives additional structure, which is what matters here. For example, we may consider the following natural elements in F_n :

Definition 2.1. *A non-crossing loop is a continuous embedding $\lambda : S^1 \hookrightarrow \mathbb{C} - \{x_1, \dots, x_n\}$ whose image contains x_0 .*

To any non-crossing loop λ , we associate the element $f_\lambda \in F_n$ obtained by following λ with the positive orientation (coming from the orientation of \mathbb{C}). Elements $f_\lambda \in F_n$ which may be obtained this are said to be non-crossing. We denote by NC the set of non-crossing elements in F_n .

We consider the length function

$$\begin{aligned} l : F_n &\longrightarrow \mathbb{Z} \\ f &\longmapsto \frac{1}{2i\pi} \sum_{j=1}^n \int_f \frac{dz}{z - x_j} \end{aligned}$$

For any non-crossing loop λ , we may consider the set $\text{Int}(\lambda)$ of points of \mathbb{C} which are “inside” λ (in the weak sense: we consider the support of λ to be “inside”). Clearly, the index of f_λ around x_i is 1 if $x_i \in \text{Int}(\lambda)$, 0 otherwise. Setting $ht(\lambda) := |\text{Int}(\lambda) \cap \{x_1, \dots, x_n\}|$, we have the relation

$$l(f_\lambda) = ht(\lambda).$$

Definition 2.2. *We define a relation \subseteq in NC by*

$$\forall f, g \in NC, f \subseteq g \stackrel{\text{def}}{\iff} \exists \text{ non-crossing loops } \lambda, \mu, f = f_\lambda, g = f_\mu, \text{Int}(\lambda) \subseteq \text{Int}(\mu).$$

We leave to the reader the following easy topological lemma:

Lemma 2.3. *For all $f, g \in NC$, the following assertions are equivalent:*

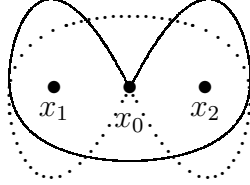
- (i) $f \subseteq g$;
- (ii) *for any non-crossing loop λ such that $f = f_\lambda$, there exists a non-crossing loop μ such that $g = f_\mu$ and $\text{Int}(\lambda) \subseteq \text{Int}(\mu)$;*
- (iii) *for any non-crossing loop μ such that $g = f_\mu$, there exists a non-crossing loop λ such that $f = f_\lambda$ and $\text{Int}(\lambda) \subseteq \text{Int}(\mu)$.*

Lemma 2.4. (i) *For all $f, g \in NC$, $f \subseteq g$ implies $l(f) \leq l(g)$. If $f \subseteq g$ and $l(f) = l(g)$, then $f = g$.*
(ii) *The relation \subseteq is an order relation.*

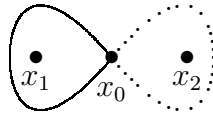
Proof. (i): The first statement is trivial. For the second statement, choose λ and μ such that $\lambda, \mu, f = f_\lambda, g = f_\mu$. Since $ht(\lambda) = ht(\mu)$, the annulus “between” λ and μ contains no point in $\{x_1, \dots, x_n\}$, thus λ and μ are isotopic.

(ii): The relation is clearly reflexive. Antisymmetry follows from (i). Transitivity follows from Lemma 2.3. \square

The main result of this section says that certain subposets of NC are lattices. Before stating it, let us observe that NC as a whole is not a lattice. A first obstruction is that one may find non-isotopic height n non-crossing loops. Clearly, they do not even have a common upper bound (let alone a least common upper bound). For $n = 2$, two such loops are illustrated below (one with a full line, the other one with a dotted line):



We may also observe that the corresponding elements in NC do not have a largest common lower bound: the two height 1 non-crossing loops represented below are distinct maximal common lower bounds to the above height 2 non-crossing loops:



Definition 2.5. *For any $g \in NC$, we set $NC_g := \{f \in NC \mid f \subseteq g\}$.*

Theorem 2.6. *For any $g \in NC$, the poset (NC_g, \subseteq) is a lattice.*

The author thanks Sang Jin Lee, for suggesting to use hyperbolic geometry in the following proof.

Proof. First, it is easy to reduce the question to the case when $l(g) = n$.

Up to isotopy, we may assume that $x_0 = -1$, and that g is represented by the unit circle. We set

$$D := \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

Using Lemma 2.3 (ii) and (iii), we observe that we may forget the outside of D : any element of $f \in NC_g$ is represented by non-crossing loops λ with $\text{Int}(\lambda) \subseteq D$, and in NC_g the relation \subseteq could be equivalently redefined using only such loops.

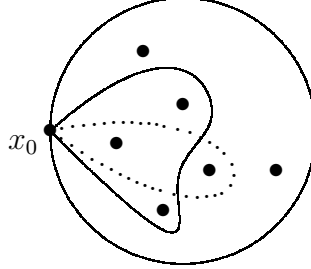
If $n = 1$, the result is straightforward.

Assume now that $n > 1$. We may endow $D_n := D - \{x_1, \dots, x_n\}$ with a complete hyperbolic metric (see, for example, [DDRW], Chapter 7) Let \widetilde{D}_n be the universal cover of D_n may be viewed as a subset of the hyperbolic plane (see the nice picture on page 114, *loc. cit.*).

Any element $f \in F_n$ may be represented by a (possibly self-intersecting) loop in the pointed space (D_n, x_0) , thus be a path in \widetilde{D}_n ; among such paths, there is a unique geodesic. The corresponding loop in (D_n, x_0) is called the *geodesic loop* of f . Geodesic loops minimise self-intersections and mutual intersections; in particular:

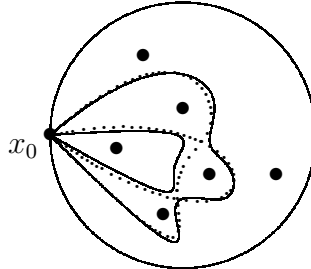
- For all $f \in F_n$, then $f \in NC_g$ if and only if its geodesic loop is non-crossing.
- For all $f, f' \in NC_g$ with geodesic loops λ, λ' , then $f \subseteq f' \Leftrightarrow \text{Int}(\lambda) \subseteq \text{Int}(\lambda')$.

The theorem is a trivial consequence of the last statement: Let $f, f' \in NC_g$ with geodesic loops λ, λ' .



Any $h \in NC_g$ such that $f \subseteq h$ and $f' \subseteq h$ may be represented by a non-crossing loop ν such that $\text{Int}(\lambda) \subseteq \text{Int}(\nu)$ and $\text{Int}(\lambda') \subseteq \text{Int}(\nu)$. Consider the loop $\lambda \vee \lambda'$ obtained by glueing the successive “outermost” portions of the two loops (in the above example, this element is made with three successive portions of loops). Clearly, any non-crossing loop containing $\text{Int}(\lambda) \cup \text{Int}(\lambda')$ in its interior must also contain $\lambda \vee \lambda'$ in its interior: the element represented by $\lambda \vee \lambda'$ is the minimal least upper bound of f and f' .

Similarly, considering the connected component of $\text{Int}(\lambda) \cap \text{Int}(\lambda')$ containing x_0 , we obtain a maximum lower bound. An illustration with the above f, f' is given below (the original loops are the dotted curves, the inf and the sup are the full curves).



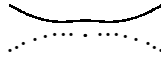
□

Remark. In the last proof, instead of using hyperbolic geometry, one could use a more computational viewpoint, which may also be used to implement the inf and sup operations. Say that two non-crossing loops are *tight* if their number of intersections is minimal

(within their homotopy classes). A first observation is that *tight* representatives for a pair of elements of NC_g may be obtained by successive “bigon eliminations”: a *bigon* is portion of the picture looking like



with no marked point x_i in the inside portion; eliminating such a bigon consists of replacing this portion of the picture by something like



Tightness may be detected by the absence of bigons. One may actually prove (by bigon elimination) the stronger result: for any triple of non-crossing loops, one may find homotopic loops which are pairwise tight. The only property of hyperbolic geodesics used above is that they are pairwise tight, thus that they solve the latter problem. However, for practical use, it is very efficient to perform bigon elimination without relying on hyperbolic geometry.

3. BRAID REFLECTIONS AND COORDINATE SYSTEMS

Since F_n is the fundamental group of the complement in \mathbb{C} of a complex algebraic hypersurface (a finite set), we may consider special elements usually called *generators-of-the-monodromy* or *meridiens* (we prefer here to call them *braid reflections*).

These elements may be described as follows. A *connecting path* is a continuous map $\gamma : [0, 1] \rightarrow \mathbb{C}$ such that $\gamma(0) = x_0$, $\gamma(1) \in \{x_1, \dots, x_n\}$ and $t \neq 1 \Rightarrow \gamma(t) \notin \{x_1, \dots, x_n\}$. One may associate to such a γ an element r_γ as follows: starting from x_0 , follow γ ; arriving close to $\gamma(1)$, make a positive turn around a small circle centered on $\gamma(1)$; return to x_0 following γ backwards.

Definition 3.1. *An element $r \in F_n$ is a braid reflection if there exists a connecting path γ such that $r = r_\gamma$. The set of reflections in F_n is denoted by R .*

Lemma 3.2. *The set R coincides with the set of non-crossing elements of height 1.*

Proof. If r is non-crossing of height 1, then choose a non-crossing loop λ representing r . We have $\text{Int}(\lambda) \cap \{x_1, \dots, x_n\} = \{x_{i_0}\}$. Since $\text{Int}(\lambda)$ is path connected, we may draw inside λ a path γ connecting x_0 and x_{i_0} . It is clear that $r = r_\gamma$.

To prove the converse statement, one may check that for any path γ connecting x_0 and some x_i , there exists $\tilde{\gamma}$ without self-intersections such that $r_\gamma = r_{\tilde{\gamma}}$ (it is clear by construction that $r_{\tilde{\gamma}}$ is non-crossing of height 1). To find such a $\tilde{\gamma}$, one may remove self-intersections by “sliding” them past x_0 . [Alternatively, one could observe that the conjugacy classes in R are indexed by the irreducible components of the hypersurface; that each conjugacy class contains a non-crossing element; and finally that NC is stable under conjugacy.] \square

The standard way to see F_n as an abstractly presented group (with n generators and no relation) is by means of a *coordinate system*:

Definition 3.3. *Consider a planar graph Γ , whose vertices are x_0, \dots, x_n , and with n edges $\gamma_1, \dots, \gamma_n$, each γ_i being a connecting path from x_0 to x_i . We assume that the γ_i ’s have no self-intersections and no mutual intersection (except at x_0).*

To each γ_i , we associate $f_i := r_{\gamma_i}$.

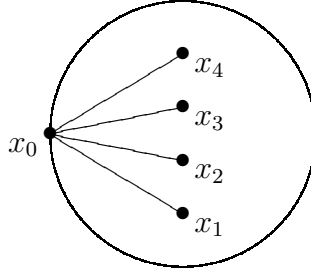
A coordinate system is the (unordered) n -tuple of reflections $\{f_1, \dots, f_n\}$ obtained this way.

We say that a coordinate system is compatible with an element $g \in NC$ if there exists a non-crossing loop γ representing g , such that Γ is drawn inside $\text{Int}(\gamma)$.

Coordinate systems are in bijection with isotopy classes of planar graphs Γ as above (isotopy with fixed vertices).

Saying that $g \in NC$ is compatible with $\{f_1, \dots, f_n\}$ is equivalent to the existence of a permutation σ such that $g = \prod_{i=1}^n f_{\sigma(i)}$. The planar structure around x_0 endows $\{f_1, \dots, f_n\}$ with a natural cyclic ordering. Once $\{f_1, \dots, f_n\}$ is fixed, choosing a compatible g is equivalent to the choice of a total ordering refining the cyclic ordering (there are n such choices).

Up to isotopy and relabelling of the marked points, we may assume that the situation looks like:

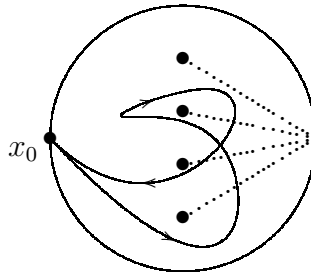


More explicitly, our assumption is that $x_0 = -1$, that the x_j are purely imaginary with

$$-1 < \Im(x_1) < \Im(x_2) < \dots < \Im(x_n) < 1,$$

and, for each j , we consider the affine connecting path $[x_0, x_j]$ and the associated braid reflection f_j . The coordinate system is then compatible with the element of NC represented by the unit circle.

We have $F_n = \langle f_1, \dots, f_n \rangle$. For any $f \in F_n$, an expression $f = \prod_{i=1}^m f_{j_i}^{\varepsilon_i}$, with $\varepsilon_i = \pm 1$, may be obtained as follows. First, find a (possibly self-intersecting) loop γ representing f and drawn inside D . Then, following γ , write f_j each time it crosses some $[x_j, 1]$ moving upwards, and f_j^{-1} each time it crosses some $[x_j, 1]$ moving downwards (up to perturbation, we may assume that γ is transversal to these segments).



In the above example, the word is $f_1 f_2 f_3^{-1} f_2^{-1}$.

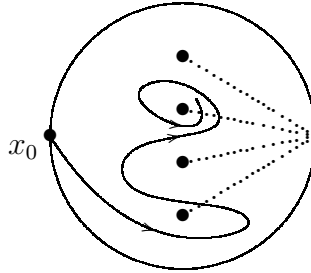
A word in the $\{f_1, f_1^{-1}, \dots, f_n, f_n^{-1}\}$ is *reduced* if the patterns $f_j f_j^{-1}$ and $f_j^{-1} f_j$ never occur. Any $f \in F_n$ admits a unique expression as a reduced word in the $\{f_1, f_1^{-1}, \dots, f_n, f_n^{-1}\}$.

A loop is *reduced* if the associated word is reduced. Clearly, any loop in D admits, in its homotopy class, a reduced loop. More precisely, this reduced loop may be obtained by a certain “bigon elimination” procedure, during which one may avoid introducing self-intersections. In particular, any non-crossing loop is homotopic to a non-crossing reduced loop.

In the next two results, we denote by g the (maximal) element of NC represented by D .

Lemma 3.4. *Let $f \in NC_g$. The reduced word associated with f is “quadrantfrei”: it does not contain the patterns $f_j f_j$ and $f_j^{-1} f_j^{-1}$.*

Proof. A picture is worth a thousand words:

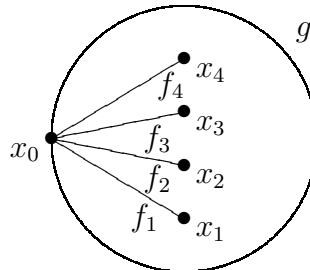


□

4. SIMPLE TRANSITIVITY OF HURWITZ ACTIONS

The material in this section is certainly classical, except the interpretation in terms of Coxeter elements in the universal Coxeter group.

Choose g a maximal non-crossing element of F_n . As we have noted earlier, it is possible to find a coordinate system f_1, \dots, f_n such that $F_n = \langle f_1, \dots, f_n \rangle$ and $g = f_1 \dots f_n$. To fix the notations, we make the standard choice for g and f_1, \dots, f_n , already used in the previous section:



Clearly, any expression of g as a product of elements of R must be of length n (consider the largest abelian quotient of F_n).

Thus

$$(f_1, \dots, f_n) \in \text{Red}_R(g).$$

Lemma 4.1. *Let $\beta \in B_n$, let $(r_1, \dots, r_n) := \beta \cdot (f_1, \dots, f_n)$. Consider a sequence of integers j_1, \dots, j_k such that $1 \leq j_1 < j_2 < \dots < j_k \leq n$. Then $r_{j_1} \dots r_{j_k} \in NC_g$.*

Proof. The elements r_1, \dots, r_n form a coordinate system (the B_n -action sends coordinate systems to coordinate systems). Up to isotopy, all coordinate systems look the same. This reduces the problem to the case when $\beta = 1$, for which the lemma is obvious. □

Definition 4.2. *The universal Coxeter group W_n is defined by the presentation:*

$$W_n := \langle s_1, \dots, s_n \mid s_i^2 = 1 \rangle.$$

We consider the epimorphism $\pi : F_n \twoheadrightarrow W_n, f_j \mapsto s_j$. We set $T := \pi(R)$. Elements of T are called reflections.

We set $c := \pi(g)$. It is again easy to see that

$$(s_1, \dots, s_n) \in \text{Red}_T(c).$$

The map $\pi^n : \text{Red}_R(g) \rightarrow \text{Red}_T(c)$ is a morphism of B_n -sets (where both sets are equipped with Hurwitz action).

Theorem 4.3. (1) *The Hurwitz action is simply transitive on $\text{Red}_R(g)$.*

(2) *The Hurwitz action is simply transitive on $\text{Red}_T(c)$.*

(3) *The map $\pi^n : \text{Red}_R(g) \rightarrow \text{Red}_T(c)$ is an isomorphism of B_n -sets.*

The author is grateful to Sang Jin Lee for pointing out that (1) was already contained in Artin's 1947 article [A].

Proof. The transitivity statement in (1) is Theorem 16 in [A] (although it appears in a formulation closer to ours at the top of p. 114 of *loc. cit.*).

Let us prove the transitivity statement in (2) – our argument is so similar to Artin's that we could have omitted the proof, but we include it for the convenience of the reader, who will easily reconstruct the proof of Artin's Theorem 16. We start with a remark about normal forms in W_n . This group is a free product of n cyclic groups of order 2. Consider a finite sequence $w := (a_1, \dots, a_m)$, where each a_i is taken in $\{s_1, \dots, s_n\}$. We say that w *represents* the element $a_1 \dots a_m \in W_n$. We say that w is the *normal form* of $a_1 \dots a_m$ if it does not contain a pattern $s_j s_j$ of consecutive equal terms. When w is a normal form, we say that m is the *length* of $a_1 \dots a_m$. Clearly, the normal form always exists and is unique. It may actually be computed with the following non-deterministic procedure. Start from an arbitrary w .

(I) If w is a normal form, return w .

(II) Otherwise, at least a pattern $s_j s_j$ appears. Choose an occurrence and remove the involved terms. Start again with the new (shorter) sequence.

A sequence of successive choices in (II) is called an *execution* of the procedure. Though there are usually several executions, the end result is always the (unique) normal form. The surviving terms in the output come from terms in the input. If we choose a particular execution, we say that a given term of w is *untouched* by the execution if it survives it.

Any $t \in T$, being a reflection, may be written

$$(*) \quad t = u_1 u_2 \dots u_k s_t u_k \dots u_2 u_1$$

where the u_i 's and s_t are in $\{s_1, \dots, s_n\}$. We may clearly assume that $(*)$ is a normal form. We say that s_t is the *content term* of t .

Let $(t_1, \dots, t_n) \in \text{Red}_T(c)$. Considering the largest abelian quotient of W_n , one may observe that, the content terms s_{t_j} satisfy $\{s_{t_1}, \dots, s_{t_n}\} = \{s_1, \dots, s_n\}$.

The normal form of (t_1, \dots, t_n) is (s_1, \dots, s_n) . Let w be the concatenation of the normal forms of t_1, \dots, t_n . Choose an execution of the normal form procedure, applied to w . The output is (s_1, \dots, s_n) . We distinguish two cases:

Case 1. The content terms of the normal forms of the t_j 's are untouched by the execution. Write

$$w = (u_1, \dots, u_k, s_1, u_k, \dots, u_1, v_1, \dots, v_l, s_2, v_l, \dots, v_1, \dots, \dots)$$

(since they are untouched, the content terms must already be in the order s_1, \dots, s_n in w). The execution rewrites w to (s_1, \dots, s_n) while leaving s_1 untouched. Thus it rewrites (u_1, \dots, u_k) to $()$. Since (u_1, \dots, u_k) is normal, this implies that $k = 0$. Considering the fragment v_1, \dots, v_l between the unaffected terms s_1 and s_2 , we conclude that $l = 0$, and so on... Thus $(t_1, \dots, t_n) = (s_1, \dots, s_n)$.

Case 2. At least one content term of one the t_j is destructed. Consider the first iteration of the execution where this happens: a certain pattern appears, involving (the descendant of) a content term of at least one of the t_j 's: denoting by (a_1, \dots, a_m) the word just before this particular iteration, we have $a_i = a_{i+1}$ for some i , with a_i or a_{i+1} being the (until then untouched) content term of one of t_j 's. Note that a_i and a_{i+1} may not both be content terms, because distinct t_j 's have distinct contents. Let us assume that a_i is the content term of some t_j . (The case when a_{i+1} is the content term may be dealt with symmetrically). Inside w , we are interested in the portion involving t_j and t_{j+1} :

$$w = (\dots, u_1, \dots, u_k, s, u_k, \dots, u_1, v_1, \dots, v_l, s', v_l, \dots, v_1, \dots),$$

where s is the content of t_j and s' the content of t_{j+1} .

Lemma 4.4. *The length of $su_k^{-1} \dots u_1^{-1} v_1 \dots v_l$ is $< l - k$ (in particular, $k < l$).*

Proof of the lemma. From the assumptions, it is easy to see that the first term s is modified in any execution with input $(s, u_k, \dots, u_1, v_1, \dots, v_l)$; in particular, this sequence is not a normal form. Consider an execution with this input.

If $k = 0$, we observe that (v_1, \dots, v_l) is a normal form. Since (s, v_1, \dots, v_l) is not normal, we must have $s = v_1$. The claim holds.

If $k > 0$, we observe that both (s, u_k, \dots, u_1) and (v_1, \dots, v_l) are normal forms. We must have u_1 and v_1 , and the first step of the execution leads to $(s, u_k, \dots, u_2, v_2, \dots, v_l)$. We conclude by an easy induction. \square

Consider the pair $(t_j t_{j+1} t_j^{-1}, t_j)$. The first reflection is represented by

$$(u_1, \dots, u_k, s, u_k, \dots, u_1, v_1, \dots, v_l, s', v_l, \dots, v_1, u_1, \dots, u_k, s, u_k, \dots, u_1).$$

By the lemma, the length of $su_k \dots u_1 v_1 \dots v_l$ is $< l - k$. The same property holds for its inverse $v_l \dots v_1 u_1 \dots u_k s$. Thus the length L of $t_j t_{j+1} t_j^{-1}$ satisfies $L < k + (l - k) + 1 + (l - k) + k = 2l + 1$. The total length of $(t_1, \dots, t_{j-1}, t_j t_{j+1} t_j^{-1}, t_j, t_{j+2}, \dots, t_n)$ is strictly smaller than the total length of (t_1, \dots, t_n) . These two decompositions lie in the same Hurwitz orbit. One may prove the transitivity part of (2) by induction on the total length.

The simplicity statement in (1) says that

$$\forall \beta \in B_n, \forall (t_1, \dots, t_n) \in \text{Red}_R(g), \beta \cdot (t_1, \dots, t_n) = (t_1, \dots, t_n) \Rightarrow \beta = 1.$$

Using the transitivity, this statement is equivalent to

$$\forall \beta \in B_n, \beta \cdot (f_1, \dots, f_n) = (f_1, \dots, f_n) \Rightarrow \beta = 1,$$

which is nothing but the faithfulness of the standard representation of B_n in $\text{Aut}(F_n)$, already known to Hurwitz.

Let us now prove the simplicity statement in (2). Using transitivity, it is enough to prove that

$$\forall \beta \in B_n, \beta \cdot (s_1, \dots, s_n) = (s_1, \dots, s_n) \Rightarrow \beta = 1.$$

Let $\beta \in B_n$ such that $\beta \cdot (s_1, \dots, s_n) = (s_1, \dots, s_n)$. Let $(r_1, \dots, r_n) := \beta \cdot (f_1, \dots, f_n)$. Since π^n commutes with Hurwitz action, we have $\pi^n((r_1, \dots, r_n)) = (s_1, \dots, s_n)$, thus $s_j = \pi(r_j)$ for all j . Fix $j \in \{1, \dots, n\}$. By Lemma 4.1, we know that $r_j \in NC_g$. Consider the normal form $f_{j_1}^{\varepsilon_1} \dots f_{j_m}^{\varepsilon_m}$ of r_j in F_n . By Lemma 3.4, this normal form is “quadratifrei”. Thus $s_{j_1} \dots s_{j_m}$ is the normal form of s_j in W_n . Thus $m = 1$ and $r_j = f_j^\varepsilon$. Since $r_j \in R$, we have $\varepsilon = 1$. This holds for any j , thus $(r_1, \dots, r_n) = (f_1, \dots, f_n)$. By (1), we must have $\beta = 1$.

(3) follows trivially. \square

Corollary 4.5. *There are natural bijections between:*

- (i) *Maximal strict chains of NC_g .*
- (ii) *Elements of $\text{Red}_R(g)$.*
- (iii) *Coordinate systems compatible with g .*

More precisely, the map from (i) to (ii) sends a maximal chain $1 = a_0 < a_1 < \dots < a_n = g$ to $(a_0^{-1}a_1, \dots, a_{n-1}^{-1}a_n)$, and the map from (ii) to (iii) send (t_1, \dots, t_n) to $\{t_1, \dots, t_n\}$.

Proof. Consider the classical interpretation of B_n as the mapping class group of the n -punctured disk, fixing the outer circle.

By Lemma 2.3, maximal strict chains of NC_g are represented by chains of concentric non-crossing loops in D , of strictly increasing height. Isotopy classes of such data clearly form a single B_n -orbit.

Similarly, coordinates systems drawn inside D form a single B_n -orbit.

The corollary then follows from the fact that $\text{Red}_R(g)$ is a single Hurwitz orbit, and that the natural maps with the above objects are B_n -equivariant. \square

Corollary 4.6. *Denote by R_g the subset of R consisting of elements which may appear in some sequence in $\text{Red}_R(g)$. Denote by T_c the subset of T consisting of elements which may appear in some sequence in $\text{Red}_T(c)$. Then $R_g = R \cap NC_g$. Moreover, π induces a bijection $R_g \simeq T_c$.*

Note that π does not induce a bijection from R to T . Also, the injectivity of $R_g \simeq T_c$ is *a priori* stronger than the injectivity of $\pi^n : \text{Red}_R(g) \rightarrow \text{Red}_T(c)$ from the theorem.

Proof. The statement $R_g = R \cap NC_g$ is already in Lemma 4.1. Using the theorem, we note that $T_c = \pi(R_g)$. We are left with having to prove the injectivity. First, we observe that the fiber of $R_g \rightarrow T_c$ over s_1 is a singleton (it follows from Lemma 3.4). By transitivity, all fibers have the same cardinal. \square

5. QUASI-GARSIDE STRUCTURE

Definition 5.1. *We denote by F_n^+ the submonoid of F_n generated by R . We endow F_n^+ with the divisibility partial ordering: for all $f, g \in F_n^+$, $f \preceq g \stackrel{\text{def}}{\iff} \exists h \in F_n^+, fh = g$.*

Note that, since R is a union of conjugacy classes, $\exists h \in F_n^+, fh = g \Leftrightarrow \exists h \in F_n^+, hf = g$. We do not have to distinguish left divisibility from right divisibility.

Lemma 5.2. *The restriction of \preceq to NC coincides with \subseteq .*

Proof. Let $f, g \in NC$.

It is constructively clear that $f \subseteq g$ implies $f \preceq g$.

Conversely, if $f \preceq g$, then a reduced R -decomposition (r_1, \dots, r_k) of f may be extended to a reduced R -decomposition (r_1, \dots, r_l) of g . By Lemma 4.1, $r_1 \dots r_k \in NC_g$. \square

In [B1, Definition 0.5.1], a Garside monoid was defined as a monoid M satisfying a certain number of axioms; one of these axioms concerns the existence of a “balanced” element $\Delta \in M$ whose set of left/right divisors is finite and generates M .

For many applications, one may work in a slightly generalised context: by *quasi-Garside monoid*, we mean a monoid satisfying all axioms of [B1, 0.5.1], except that we do not require the set of divisors of Δ to be finite.

Theorem 5.3. *Let g be a maximal element of NC . Let M_g be the submonoid of F_n generated by $\{r \in R \mid r \preceq g\}$. Then M_g is a quasi-Garside monoid with Garside element g and set of simples NC_g .*

Proof. Set $P_g := \{r \in R \mid r \preceq g\}$. Using a straightforward analog of [B1, Theorem 0.5.2] where the finiteness condition is removed, we only have to prove that (P_g, \preceq) is a lattice.

Using the last lemma, we see that any element of NC_g lies in P_g ; conversely, using Corollary 4.5, we see that any element of P_g belongs to NC_g ; using again the last lemma, we have $(P_g, \preceq) = (NC_g, \subseteq)$. By Theorem 2.6, the latter is a lattice. \square

The free group being easy enough to study with the classical point of view (with its presentation with n generators and 0 relations) that what brings the above quasi-Garside structure may seem futile: for example, we have a new presentation with an infinity of generators (reflections in NC_g) and an infinity of relations of length 2 (the relations $rr' = r''r$, whenever $r, r' \in NC_g$ satisfy $rr' \in NC_g$ and $r'' = rr'r^{-1}$), with a solution to the word and conjugacy problem... The main interest of this quasi-Garside structure is that it fits in a general pattern, formalised in the conjectures below, and also that it is useful to understand geometric aspects of complex reflection groups, as it will appear in the sequel [B2].

6. CONJECTURES

As announced in the introduction, our conjectures apply to two different settings:

- (a) either (W, S) is a Coxeter system; we assume that $n := |S|$ is finite (but W may be infinite); we denote by T the set of reflections in W (arbitrary conjugates in W of elements of S); we consider the associated Artin group $B := A(W, S)$ (we will use bold fonts to refer to the formal copy of S generating B); we denote by R the set of “braid reflections” (arbitrary conjugates in B of elements of \mathbf{S});
- (b) or W is an irreducible complex reflection group of rank n generated by involutive reflections; we assume that it is “well-generated”, *i.e.*, it may be generated by n reflections; we denote by T the set of all reflections in W ; we consider the generalised braid group $B := B(W)$, defined in [BMR] as the fundamental group of the

space of regular orbits; we denote by R the set of “braid reflections” (“generators-of-the-monodromy”) in B .

In both settings, there is a natural map $p : B \twoheadrightarrow W$.

Definition 6.1. *A Coxeter element is, depending on the setting:*

- (a) *the conjugate in W of a product $s_1 \dots s_n$, for a certain numbering $S = \{s_1, \dots, s_n\}$;*
- (b) *an element $c \in W$ such that $\ker(c - e^{\frac{2i\pi}{d_n}}) \neq 0$, where d_n is the largest invariant degree of W .*

A braid Coxeter element is, depending on the setting:

- (a) *the conjugate in B of a product $\mathbf{s}_1 \dots \mathbf{s}_n$, for a certain numbering $\mathbf{S} = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$;*
- (b) *an element $g \in B$ such that $g^{d_n} = \pi$, where d_n is the largest invariant degree of W , and π is the standard “full-turn” element in the center of B ([BMR]).*

Clearly, in the situation (a), p maps braid Coxeter elements to Coxeter elements. This also holds in (b) ([B2]).

An important issue is that, in situation (a), there are usually several conjugacy classes of (braid) Coxeter element. However, when the Coxeter graph is a tree, there is a unique conjugacy class ([LIE], p. 117). In the situation of the free group, there are many conjugacy classes, but they are group-theoretically undistinguishable, since the full symmetric group acts by diagram automorphisms.

In our conjectures, only the conjugacy class of the braid Coxeter element matters.

Conjecture 6.2. *There exists a braid Coxeter element $g \in B$ such that, setting $c := p(g)$, we have:*

- (1) *The Hurwitz action is transitive on $\text{Red}_R(g)$.*
- (2) *The Hurwitz action is transitive on $\text{Red}_T(c)$.*
- (3) *The map p^n induces an isomorphism of B_n -sets from $\text{Red}_R(g)$ to $\text{Red}_T(c)$.*
- (4) *The map p induces a bijection from the set R_g of reflections appearing in $\text{Red}_R(g)$ to the set T_c of braid reflections appearing in $\text{Red}_T(c)$.*

In the case of the universal Coxeter group W_n and its braid group F_n , the conjecture is proved above (Theorem 4.3 and Corollary 4.6). That the action is then *simply* transitive and not just transitive is specific to this case.

When W is a finite Coxeter group, most of the conjecture is proved in [B1]: (2) is *loc. cit.* Proposition 1.6.1, and a weaker form of (3) and (4) are consequences of Fact 2.2.4; however, no description of $\text{Red}_R(g)$ is given (only a specific B_n -orbit is considered, it is not proved to be the full $\text{Red}_R(g)$).

When W is the Coxeter group of type \widetilde{A}_n , this follows from [D, Proposition 3.4]. Note that Digne proves a more general result: the transitivity is true for all braid Coxeter elements. The above conjecture is certainly not optimal (see for example Digne’s Conjecture 1.1). Actually, in view of [Br, Theorem 3.16] (and the discussion following this result on p. 87), it is tempting to formulate a more general conjecture, not only applying to Coxeter elements but to elements whose reduced decompositions involve generating sets. However, since we have neither interesting examples nor applications, we stay with the above conjecture, which interests us in connection with our second conjecture below.

Given any braid Coxeter element g , consider the positive presentation with set of generators R_g and relations $rr' = r''r$ whenever there exists an element of $\text{Red}_R(g)$ starting by (r, r', \dots) and $r'' = rr'r^{-1}$. Let M_g be the monoid defined by this presentation; let B_g be the group defined by this presentation.

Since the relations $rr' = r''r$ hold in B , B is *a priori* a quotient of B_g . In setting (a), it is easy to see that the defining relations of B are consequences of the Hurwitz relations, thus that $B_g \simeq B$. One may prove the similar statement in setting (b) ([B2]).

Points (1) and (2) of the above conjecture express that M_g coincides with the monoids associated with the triples (B, R, g) and (W, T, c) , as in [B1, Section 0.4]. With the obvious analog of [B1, Theorem 0.5.2], the next conjecture is the key ingredient to prove that M_g is a quasi-Garside monoid.

Conjecture 6.3. *Denote by B_+ the submonoid of B generated by R . Denote by \preceq the relation on B_+ defined by $b \preceq b'$ if and only if $b^{-1}b' \in B_+$. For any $b \in B_+$, set $P_b := \{b' \in B_+ | b' \preceq b\}$. There exists a braid Coxeter element $g \in B$ satisfying Conjecture 6.2 and such that (P_g, \preceq) is a lattice.*

Again, this is known for spherical Artin types, [B1], and affine type \tilde{A} , [D], and in F_n as it was proved above. The most mysterious aspect is that the lattice does *not* hold for all Coxeter elements: indeed, Digne's striking Proposition 5.5 shows that, in $\widetilde{A_{n-1}}$, it holds only when the braid Coxeter element is a product of the generators according to the cyclic order on the diagram. We have no good hint on how to characterise suitable braid Coxeter elements in setting (a). In setting (b), all choices are conjugate.

Among possible applications, we observe that braid groups satisfying conjectures 6.2 and 6.3 have cohomological dimension smaller or equal to n , since the construction of [CMW] of a simplicial $K(\pi, 1)$ for Garside groups clearly extends to quasi-Garside groups (the obtained $K(\pi, 1)$ still being of dimension n , but no longer necessarily finite).

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